Dynamical Systems

Exam 12 April 2024

university of groningen

The exam consists of 4 questions. You have 120 minutes to do the exam. You can achieve 50 points in total which includes a bonus of 5 points.

1. [**3+3+3=9 Points**]

   For each of the following bifurcations, plot the bifurcation diagram, describe in words the bifurcation scenario, and give explicit examples of a time continuous system showing the respective bifurcation.

   (a) Saddle-node bifurcation

   (b) Pitchfork bifurcation

   (c) Hopf bifurcation

2. [**10 Points**]

   Consider the planar system

   $$X' = \begin{pmatrix} 2a & b \\ -b & 0 \end{pmatrix} X\,.$$

   Sketch the regions in the $a-b$ plane where this system has different types of canonical forms. In each region give the canonical form and sketch the phase portrait of the system in canonical form.

3. [**2+4+3+5=14 Points**]

   Consider the pendulum

   $$x' = y,$$
   $$y' = -\nu y - \sin x,$$

   where $\nu \geq 0$ is a friction parameter and $(x, y) \in S^1 \times \mathbb{R}$ where we view the circle S^1 to be the interval $[-\pi, \pi]$ with the boundaries identified.

   (a) Show that the system has two equilibrium points located at $(x_0, y_0) := (0, 0)$ and $(x_1, y_1) := (\pm\pi, 0)$.

   (b) Show from the linearization that the equilibrium point at (x_1, y_1) is a saddle and for $\nu > 0$, the equilibrium point at $(x_0, y_0) = (0, 0)$ is a sink.

   (c) Show that for $\nu = 0$, the system is Hamiltonian by constructing a Hamilton function H.

   (d) Show that for $\nu \geq 0$ and $0 < h < 2$, $H(x, y) - H(0, 0)$ is a Lyapunov function for the equilibrium at (x_0, y_0) in the region $D_h = \{(x, y) \in S^1 \times \mathbb{R} \,|\, H(x, y) - H(0, 0) \leq h,\ |x| \leq \pi\}$ where H is the Hamilton function from part (c). Use the Lasalle Invariance Principle to show that for $\nu > 0$, the equilibrium at $(x_0, y_0) = (0, 0)$ is asymptotically stable with D_h belonging to the basin of attraction.

– **please turn over** –

4. [3+9 **Points**]

(a) State the three conditions of Devaney's definition of chaos for a discrete time system defined on an interval in $\mathbb{R}$.

(b) Let $\Sigma = \{(s_0, s_1, s_2, \ldots) : s_k \in \{0, 1\}\}$ be the space of half-infinite sequences of the symbols 0 and 1. The map $d : \Sigma \times \Sigma \to \mathbb{R}$ which maps $s = (s_0, s_1, s_2, \ldots)$ and $t = (t_0, t_1, t_2, \ldots)$ to

$$d(s,t) = \sum_{k=0}^{\infty} \frac{|s_k - t_k|}{2^k}$$

defines a metric on Σ. Prove that the shift map

$$\sigma : \Sigma \to \Sigma, \quad s = (s_0, s_1, s_2, \ldots) \mapsto \sigma(s) = (s_1, s_2, s_3, \ldots)$$

is chaotic by showing that all three conditions of Devaney's definition of chaos are satisfied.